On the Matrix which Represents a Vector.

By C. H. Chapman.

The fundamental idea of this work is emphasized by Buchheim as follows: "It is, or ought to be, well known that the linear and vector function of a vector is simply the matrix of the third order,"* meaning, of course, that the matrix is the operator which transforms ρ into $\phi\rho$ (Tait's Quaternions, p. 98. I shall refer to Tait's work from time to time simply as "Tait," the edition being that of 1867). It is merely a very special application of this remark to observe that any vector whatever whose rectangular components have the lengths x, y, z may be derived from that of which the components are of length 1, 1, 1 by the operation of the matrix

$$\left(\begin{array}{ccc} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{array}\right|,$$

and it follows that the study of vectors may be made to depend upon that of these matrices. The object of the present paper is to carry out and illustrate this study in some detail.

In the course of the work the matrix is freely spoken of as the vector by an allowable looseness of language, since no confusion can result; but it is not assumed that they are the same—in fact I wish to guard most carefully against that assumption.

The algebra of these matrices is the same as that of scalar quantities, inasmuch as, owing to their very special form, they are commutative with one another in multiplication; for this reason I have found it necessary to employ only a few very elementary properties of matrices. In fact Cayley's celebrated memoir (Phil. Trans. 1858) contains nearly all the theorems which are here

* Proc. London Math. Soc., Vol. XVI, p. 63; Clifford, Dynamic, p. 186.

made use of, although I have referred by preference to the papers by Sylvester (*Am. Journ. Math.*, Vol. VI) and Taber (*Am. Journ. Math.*, Vol. XII) as being more recently in the hands of all readers. The learned article of Taber contains a statement of pretty nearly the entire theory of matrices, at least from his point of view. The only new symbol here introduced is that of a circular substitution, *cy* or *cy'*, which has proved very interesting and useful; for the rest, the symbols S and V, not as selective but merely appellative, but with their geometrical significance essentially unchanged, have been found sufficient.

Many, if not all, the advantages which result from the use of quaternions will be found attained or attainable in this paper without their aid by the use of these matrices; the equations and formulas are sometimes identical in appearance with those of the quaternion calculus, and the resemblance has been systematically cherished—partly to aid in making comparisons, partly on account of the extreme elegance attained by the quaternion calculus in the hands of Hamilton and Tait.

The unpublished Vector Analysis of J. Willard Gibbs is briefly referred to in this article as V. A. G.

1.—*Properties of the Matrix.*

The matrix $\begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix} = \rho$ is referred to its axes,* which will be assumed as a rectangular system and, for distinction, designated as the axes of i, j and k respectively. The matrix ρ denotes that substitution which transforms the vector whose components along the three axes have the lengths 1, 1, 1 into that whose components are of lengths x, y, z respectively. The matrix therefore defines without ambiguity the length and direction of this vector, and I shall say that the matrix is the analytical expression or representation of the vector.

In particular the matrix $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ represents a vector whose length is $\sqrt{3}$ and which makes equal angles with the positive directions of the axes. In the

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* Taber, *Am. Journ. Math.*, Vol. XII, p. 360.

vector theory this matrix plays the part of unity in the scalar and will be spoken of as vector unity and denoted by v.*

For brevity, matrices of the kind here considered will be denoted by Greek letters, and the same letter will uniformly denote both the matrix and the vector represented by it. If

$$\alpha = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \text{ and } \beta = \begin{vmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{vmatrix},$$

then

$$\alpha + \beta = \begin{vmatrix} a + a', & 0 \;, & 0 \\ 0 \;, & b + b', & 0 \\ 0 \;, & 0 \;, & c + c' \end{vmatrix}^{\dagger}.$$

Whence it is evident that the sum of the matrices represents the sum of the vectors.‡

The product of the matrices α and β is

$$\alpha\beta = \begin{vmatrix} aa' & 0 & 0 \\ 0 & bb' & 0 \\ 0 & 0 & cc' \end{vmatrix},$$

which is evidently the same as $\beta\alpha$, and, for the purposes of this paper, the product of the vectors α and β will be so defined that it may be represented by the matrix $\alpha\beta$; that is, this product is a new vector whose component lengths are the products of those of the factors. The writer is well aware of his presumption in thus departing from illustrious usage,§ but inasmuch as he thus attains analytical consistency, simplicity of notation, and preservation of the associative law, both of which latter are lacking in the ingenious work of J. Willard Gibbs, and at the same time retains the freedom of commutative multiplication, the lack of which makes Hamilton's system at once so significant, so fascinating, and so difficult for ordinary students to master and apply, he has ventured, in a tentative way, to make the departure. Unless all the constituents of the matrix except those of the principal diagonal are zeros, $\alpha\beta$ is not in general the same as $\beta\alpha$, so that I by no means imply by this that linear vector functions are in general commutative in multiplication.

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* Sylvester, *Am. Journ. Math.*, Vol. VI, pp. 274 and 275.

† Sylvester, *Am. Journ. Math.*, Vol. VI, pp. 274 and 275.

‡ V. A. G., p. 4.

§ Tait, p. 44 ff.; V. A. G., p. 5.

Since division of matrices is defined by the equation

$$\beta \frac{\alpha}{\beta} = \alpha,^*$$

we see at once that the quotient of the matrix α by β is the matrix

$$\begin{pmatrix} \frac{a}{a'} & 0 & 0 \\ 0 & \frac{b}{b'} & 0 \\ 0 & 0 & \frac{c}{c'} \end{pmatrix}$$

which will be taken to represent the quotient of the vector α by the vector β. From this it follows that the division is not ambiguous and that

$$\beta \frac{\alpha}{\beta} = \frac{\alpha}{\beta} \beta = \alpha.$$

It is to be observed that this division is indeterminate or gives a quotient with an infinite constituent in cases quite analogous to those of scalar division.†

Without using Sylvester's Theorem,‡ it is evident from the rules for multiplication and division that any function $f(\rho)$ which can be expanded in positive or negative powers of ρ will be denoted by the matrix

$$\begin{pmatrix} f(x) & 0 & 0 \\ 0 & f(y) & 0 \\ 0 & 0 & f(z) \end{pmatrix}.$$

The matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will be denoted by i, j, k respectively. For their multiplication we have

$$i^2 = i, \quad j^2 = j, \quad k^2 = k, \quad ij = ik = jk = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = ijk.$$

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* Sylvester, *Am. Journ. Math.*, Vol. VI, p. 276.

† V. A. G., p. 47, Art. 128; Sylvester, *Am. Journ. Math.*, Vol. VI, p. 274.

‡ Quoted and verified by Taber, p. 378; Buchheim, Proc. Lond. Math. Soc., Vol. 16, p. 81.

Also $\quad i\rho = xi;\ j\rho = yj;\ k\rho = zk;$

whence $\quad (i+j+k)\rho = v\rho = \rho = i\rho + j\rho + k\rho;$

being a formula for decomposing a vector into three rectangular components.

The matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I,\ J,\ K,$$

have the properties that

$$\begin{gathered} I^2 = I,\ J^2 = J,\ K^2 = K, \\ IJ = JI = j;\ IK = i;\ JK = k, \\ iI = i;\ iJ = 0;\ iK = i, \\ jI = j;\ jJ = j;\ jK = 0, \\ kI = 0;\ kJ = k;\ kK = k. \end{gathered}$$

The vectors i, j, k; I, J, K cannot be used as divisors.

2.—*The Operators cy and cy′ and the Symbols S and V.*

If ρ denote the matrix $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, or briefly (x, y, z), then shall $cy\rho$ denote (y, z, x) and $cy^2\rho$, (z, x, y); cy being an abbreviation for the word cyclic and denoting evidently a circular substitution among the constituents of ρ.* Clearly also

$$cy^3\rho = \rho.$$

Again, if α, β, γ are any three rectangular unit vectors and $\rho = x\alpha + y\beta + z\gamma$, cy' shall denote the substitution

$$(\alpha, \beta, \gamma;\ \gamma, \alpha, \beta),$$

so that $\quad cy'\rho = x\gamma + y\alpha + z\beta;\ cy'^2\rho = x\beta + y\gamma + z\alpha,\ cy'^3\rho = \rho.$

Denoting $\alpha + \beta + \gamma$ by v', cy and cy' are operators which rotate ρ in a negative direction about the axes v and v' respectively, through an angle equal to $\frac{2\pi}{3}$, while the rotation performed by cy^2 and cy'^2 is $\frac{4\pi}{3}$.

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* Cf. Taber, *Am. Journ. Math.*, Vol. XIII, p. 162.

If $\alpha = (a, b, c)$ and $\beta = (a', b', c')$ are any two matrices of the kind in question, then $\alpha\beta = (aa', bb', cc')$ and

$$S.\alpha\beta = aa' + bb' + cc'$$

will be called the scalar of $\alpha\beta$. It is the product of the lengths of the vectors into the cosine of their included angle, and has thus its usual geometrical meaning. Its practical use is the same as in quaternions. Since in any product the vector v may be suppressed, the expression $S.v\rho = x + y + z$ will often be written simply

$$S.\rho.$$

Obviously $\quad cyS\rho = Scy\rho$ and $cyS.\alpha\beta = Scy(\alpha\beta)$.

Also $\quad cy(\alpha, \beta, \gamma, \delta \ldots.) = cy\alpha\, cy\beta\, cy\gamma \ldots.$

and $\quad cy\dfrac{\alpha}{\beta} = \dfrac{cy\alpha}{cy\beta}$.

While $\quad cy(\alpha cy\beta) = cy\alpha cy^2\beta$ and $S.\alpha cy^2\beta = S.\beta cy\alpha$,

from the definitions of the symbols. We recall also that if $S.\alpha\beta = 0$, α and β are perpendicular to each other, and note that the product of any two vectors perpendicular to each other is a third vector perpendicular to vector unity.

In particular,

$$cyi = k\ , \quad cyk = j\, , \quad cyj = i\, ,$$
$$cyI = K, \quad cyK = J, \quad cyJ = I.$$

Let us now form the vector

$$\delta = cy\alpha\, cy^2\beta - cy\beta\, cy^2\alpha.$$

We have by the above formulas

$$S.\alpha\,(cy\alpha\, cy^2\beta - cy\beta\, cy^2\alpha) = S.\alpha cy\alpha\, cy^2\beta - S.\alpha cy\alpha\, cy^2\beta = 0;$$

and likewise $\quad S.\beta\,(cy\alpha\, cy^2\beta - cy\beta\, cy^2\alpha) = 0.$

Hence δ is perpendicular to α and β. Its direction is such that a positive rotation about it as an axis turns α to β, and its length is the product of the lengths of α and β into the sine of their included angle; it is therefore the vector which Hamilton denotes by $V\alpha\beta$; but since we do not derive it from α and β by multiplication, it will be denoted in this paper by $V(\alpha, \beta)$, or $-V(\beta, \alpha)$.

We have $\quad V(i, j) = k;\ \ V(j, k) = i,\ \ V(k, i) = j.$

Also, if α, β, γ are any rectangular system of unit vectors,

$$V(\alpha, \beta) = \gamma; \; V(\gamma, \alpha) = \beta; \; V(\beta, \gamma) = \alpha.$$

We observe now that taking $\alpha + \beta + \gamma = v'$ for the axis of cy', then

$$cy'\alpha = \gamma, \; cy'\beta = \alpha, \; cy'\gamma = \beta;$$

and that if $\rho = x\alpha + y\beta + z\gamma$, then

$$S.v'\rho = x + y + z.$$

Finally, however ρ may be expressed, the square of its length is

$$S.v\rho^2 = S.\rho^2.$$

It will now be clear that

$$S.\alpha V(\beta, \gamma) = S.\beta V(\gamma, \alpha) = S.\gamma V(\alpha, \beta),^*$$

whatever vectors α, β and γ may be; and that if $S.\alpha V(\beta, \gamma) = 0$, the three vectors lie in the same plane.

The coefficients x, y and z in the expression

$$\delta = x\alpha + y\beta + z\gamma,$$

where α, β, γ do not lie in the same plane, can now be determined. Multiplying by $V(\alpha, \beta)$ and taking the scalars, we find

$$z = \frac{S.\delta V(\alpha, \beta)}{S.\gamma V(\alpha, \beta)} \tag{9}$$

and similarly

$$x = \frac{S.\delta V(\beta, \gamma)}{S.\alpha V(\beta, \gamma)}; \; y = \frac{S.\delta V(\gamma, \alpha)}{S.\beta V(\gamma, \alpha)}.$$

Whence

$$\delta S.\alpha V(\beta, \gamma) = \alpha S.\delta V(\beta, \gamma) + \beta S.\delta V(\gamma, \alpha) + \gamma S.\delta V(\alpha, \beta).\dagger \tag{1}$$

This expression is scarcely less simple than the corresponding one in quaternions, from which it differs in form only by the notation $V(\alpha, \beta)$ instead of $V.\alpha\beta$. In the quaternion formula $S.(\alpha V.\beta\gamma)$ the V may be omitted without changing the meaning, but from $S.\alpha V(\beta, \gamma)$ it cannot.

It is also possible to expand δ in the form

$$\delta = xV(\gamma, \alpha) + yV(\alpha, \beta) + zV(\beta, \gamma).$$

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* Cf. Tait, pp. 56 and 65. † Tait, p. 57.

We have

$$z = \frac{S.\alpha\delta}{S.\alpha V(\beta, \gamma)}; \; y = \frac{S.\gamma\delta}{S.\gamma V(\alpha, \beta)}; \; x = \frac{S.\beta\delta}{S.\beta V(\gamma, \alpha)}.$$

Hence

$$\delta S.\alpha V(\beta, \gamma) = V(\gamma, \alpha)\, S.\beta\delta + V(\alpha, \beta)\, S.\gamma\delta + V(\beta, \gamma)\, S.\alpha\delta. \tag{2}$$

The following formulas which follow directly from the definitions of the symbols cy and S are useful in what follows:

$$\left.\begin{aligned} vS.\rho &= \rho + cy\rho + cy^2\rho, \\ vS.\rho^2 &= \rho^2 + cy\rho^2 + cy^2\rho^2, \\ v(S^2.\rho - S.\rho^2) &= 2vS.\rho cy\rho. \end{aligned}\right\} \tag{3}$$

whence

Also

$$v'S.v'\rho = \rho + cy'\rho + cy'^2\rho. \tag{4}$$

The cosine of the angle which ρ makes with v is

$$\frac{S.\rho}{\sqrt{3S.\rho^2}},$$

and the cosine of the angle between ρ and v' is

$$\frac{S.v'\rho}{\sqrt{3S.\rho^2}}.$$

The components of ρ parallel and perpendicular to v are respectively

$$\tfrac{1}{3}vS.\rho \text{ and } \rho - \tfrac{1}{3}v.S\rho; \tag{5}$$

while the same with respect to v' are

$$\tfrac{1}{3}v'S.v'\rho \text{ and } \rho - \tfrac{1}{3}v'S.v'\rho. \tag{6}$$

3.—*The Symbols* $cy^{\frac{3\vartheta}{2\pi}}$ *and* $cy'^{\frac{3\vartheta}{2\pi}}$.

Since cy^3 and cy'^3 perform respectively negative rotations equal to 2π about their respective axes, it is very natural to indicate corresponding rotations through the angle ϑ by $cy^{\frac{3\vartheta}{2\pi}}$ and $cy'^{\frac{3\vartheta}{2\pi}}$. When $\vartheta = \frac{2\pi}{3}$,

$$cy^{\frac{3\vartheta}{2\pi}} = cy, \; cy'^{\frac{3\vartheta}{2\pi}} = cy';$$

when $\vartheta = \frac{4\pi}{3}$,

$$cy^{\frac{3\vartheta}{2\pi}} = cy^2; \; cy'^{\frac{3\vartheta}{2\pi}} = cy'^2.$$

I propose to find a matrix μ such that

$$\mu\rho = cy^{\frac{3\vartheta}{2\pi}}\rho.$$

Evidently for $\vartheta = \frac{2\pi}{3}$,

$$\mu = \begin{pmatrix} \frac{y}{x} & 0 & 0 \\ 0 & \frac{z}{y} & 0 \\ 0 & 0 & \frac{x}{z} \end{pmatrix} \tag{7}$$

and for $\vartheta = \frac{4\pi}{3}$,

$$\mu = \begin{pmatrix} \frac{z}{x} & 0 & 0 \\ 0 & \frac{x}{y} & 0 \\ 0 & 0 & \frac{y}{z} \end{pmatrix}. \tag{8}$$

We observe that ρ and $\mu\rho$ have the same length, and make the same angles with v; hence we obtain the equations

$$S.\mu\rho = S.\rho; \tag{9}$$

$$S.\mu^2\rho^2 = S.\rho^2. \tag{10}$$

To obtain the third necessary scalar equation, we note that ϑ is the angle between $\frac{1}{3}(3\rho - vS.\rho)$ and $\frac{1}{3}(3\mu\rho - vS.\mu\rho)$, the vector projections of ρ and $\mu\rho$ respectively perpendicular to v. Hence

$$\cos\vartheta = \frac{\frac{1}{9}S.(3\rho - vS.\rho)(3\mu\rho - vS.\mu\rho)}{\frac{1}{9}S.(3\rho - vS.\rho)^2} = \frac{3S.\mu\rho^2 - S.\rho S.\mu\rho}{3S.\rho^2 - S^2.\rho}.$$

Whence

$$3S.\mu\rho^2 = (3S.\rho^2 - S^2.\rho)\cos\vartheta + S^2.\rho. \tag{11}$$

Instead of solving these three equations directly for μ, let us approach the result indirectly by aid of a vector ν which rotates $\mu\rho$ through an angle ϕ, leading to the equations

$$S.\nu\mu\rho = S.\mu\rho = S.\rho, \tag{12}$$

$$S.\nu^2\mu^2\rho^2 = S.\mu^2\rho^2 = S.\rho^2, \tag{13}$$

$$3S.\nu\mu^2\rho^2 = (3S.\rho^2 - S^2.\rho)\cos\phi + S^2.\rho. \tag{14}$$

Again we notice that $\nu\mu$ is a vector which rotates ρ about v through the angle $\vartheta + \phi$ without altering its length, and thus deduce the additional equations

$$3S.\nu\mu\rho^2 = (3S.\rho^2 - S^2.\rho)\cos(\vartheta + \phi) + S^2\rho. \tag{15}$$

Now in general the vectors $\mu\rho$, $\mu\rho^2$, and $\mu^2\rho^2$ do not lie in the same plane, for we have

$$vS.\mu^2\rho^2 V(\mu\rho,\, \mu\rho^2) = vS.\mu^2 cy\mu cy^2\mu\rho^2 cy\rho cy^2\rho\,(cy^2\rho - cy\rho)\,, \tag{16}$$

remembering that $cy\rho^2 = (cy\rho)^2$ and $cy^2\rho^2 = (cy^2\rho)^2$. And if we take into account the fact that

$$vS.\rho cy\rho cy^2\rho\sigma = \rho cy\rho cy^2\rho S.\sigma \tag{17}$$

we may write equation (16)

$$vS.\mu^2\rho^2 V(\mu\rho,\, \mu\rho^2) = \mu cy\mu cy^2\mu\rho cy\rho cy^2\rho S.\mu\rho\,(cy^2\rho - cy\rho)\,. \tag{18}$$

If $S.\mu\rho\,(cy^2\rho - cy\rho) = 0$, then μ satisfies a fourth scalar equation besides (9), (10) and (11). This can happen, for a given value of ρ, only in very special cases, and one such case is $\mu = v$.

In general, then, by aid of equation (2), we may write

$$\begin{aligned} &\nu.S.\mu\rho V(\mu\rho^2,\, \mu^2\rho^2) \\ &\quad = V(\mu^2\rho^2,\, \mu\rho)\, S.\nu\mu\rho^2 + V(\mu\rho,\, \mu\rho^2)\, S.\nu\mu^2\rho^2 + V(\mu\rho^2,\, \mu^2\rho^2)\, S.\nu\mu\rho\,. \end{aligned} \tag{19}$$

In simplifying the formula (18) it will be useful to note that

$$V(\delta\alpha,\, \delta\beta) = cy\delta cy^2\delta . V(\alpha,\, \beta)\,. \tag{20}$$

We may see this to be true since

$$\begin{aligned} V(\delta\alpha,\, \delta\beta) &= cy\,(\delta\alpha)\, cy^2\,(\delta\beta) - cy\,(\delta\beta)\, cy^2\,(\delta\alpha) \\ &= cy\delta cy^2\delta\,(cy\alpha cy^2\beta - cy\beta cy^2\alpha) = cy\delta cy^2\delta . V(\alpha,\, \beta)\,. \end{aligned}$$

Using this formula, equation (19) becomes

$$\begin{aligned} \nu S.\mu\rho cy\,(\mu\rho^2)\, cy^2\,(\mu\rho^2)\, V(v,\, \mu) &= cy\,(\mu\rho)\, cy^2\,(\mu\rho)\, V(\mu\rho,\, v)\ S.\nu\mu\rho^2 \\ &+ cy\,(\mu\rho)\, cy^2\,(\mu\rho)\, V(v,\, \rho)\, S.\nu\mu^2\rho^2 + cy\,(\mu\rho^2)\, cy^2\,(\mu\rho^2)\, V(v,\, \mu)\, S.\nu\mu\rho\,. \end{aligned} \tag{21}$$

Using the formula (17) we see that this equation is divisible by the factor $cy\mu\rho cy^2\mu\rho$, giving for the quotient

$$\begin{aligned} &\nu\mu\rho S.cy\rho cy^2\rho\, V(v,\, \mu) \\ &\quad = V(\mu\rho,\, v)\, S.\nu\mu\rho^2 + V(v,\, \rho)\, S.\nu\mu^2\rho^2 + cy\rho cy^2\rho\, V(v,\, \mu)\, S.\nu\mu\rho\,. \end{aligned} \tag{22}$$

To complete the determination of ν we shall assign to μ a particular value such that knowing the effect of ν on $\mu\rho$, we can readily find the vector which rotates ρ through the same angle. The vector $\mu = \dfrac{cy\rho}{\rho}$ satisfies equations (9),

(10) and (11) but not (18), except in the special cases when ρ is the vector to the surface
$$xy + yz + zx = x^2 + y^2 + z^2.$$

We may therefore take $\mu = \frac{cy\rho}{\rho}$, and the corresponding value of ϑ is $\frac{2\pi}{3}$. With these values equation (22) becomes

$$3\nu cy\rho S.cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) = V(cy\rho, v)\left[(3S.\rho^2 - S^2.\rho)\cos\left(\frac{2\pi}{3} + \phi\right) + S^2.\rho\right]$$
$$+ V(v, \rho)[(3S.\rho^2 - S^2.\rho)\cos\phi + S^2.\rho] + 3cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right)S.\rho. \quad (23)$$

Now
$$cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) = cy\rho cy^2\rho\left(\frac{cy^3\rho}{cy^2\rho} - \frac{cy^2\rho}{cy\rho}\right) = \rho cy\rho - cy^2\rho^2,$$
and
$$S.(\rho cy\rho - cy^2\rho^2) = S.\rho cy\rho - S.\rho^2;$$
since
$$S.cy^2\rho^2 = cy^2S.\rho^2 = S.\rho^2.$$

And from equations (3),
$$S.\rho cy\rho = \tfrac{1}{2}(S^2.\rho - S.\rho^2).$$

Hence
$$S.cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) = \tfrac{1}{2}(S^2.\rho - 3S.\rho^2).$$

Again,
$$\begin{aligned}
&[V(cy\rho, v) + V(v, \rho)]\, S^2.\rho + 3cy\rho cy^2\rho V\left(v, \frac{cy\rho}{\rho}\right) S.\rho \\
&\quad = V(v, \rho - cy\rho)\, S^2.\rho + 3V(\rho, cy\rho)\, S.\rho \\
&\quad = [V(\rho + cy\rho + cy^2\rho, \rho - cy\rho) + 3V(\rho, cy\rho)]\, S.\rho \\
&\quad = [V(\rho, cy\rho) + V(cy^2\rho, \rho - cy\rho)]\, S.\rho = \tfrac{1}{2}v\,[S^2.\rho - 3S.\rho^2]\, S.\rho.
\end{aligned}$$

Dividing out the common factor $[S^2.\rho - 3S.\rho^2]$ from equation (23) we have left
$$\tfrac{3}{2}\nu cy\rho = -V(cy\rho, v)\cos\left(\frac{2\pi}{3} + \phi\right) - V(v, \rho)\cos\phi + \tfrac{1}{2}v\, S.\rho,$$
or finally,
$$3\nu cy\rho = 2V(v, cy\rho)\cos\left(\frac{2\pi}{3} + \phi\right) + 2V(\rho, v)\cos\phi + vS.\rho. \quad (24)$$

Changing $cy\rho$ to ρ and consequently ρ to $cy^2\rho$, and recalling that
$$S.\rho = S.cy\rho = S.cy^2\rho,$$

we have, if we replace ν by μ,

$$3\mu\rho = 2V(v, \rho)\cos\left(\frac{2\pi}{3} + \phi\right) + 2V(cy^2\rho, v)\cos\phi + vS.\rho. \tag{25}$$

Now, inasmuch as ν was a vector which rotated $cy\rho$ through the angle ϕ, it must be that μ has the same effect on ρ.

Equation (25) is homogeneous in the tensor of ρ and μ is therefore not a function of the length of ρ, but depends merely on its direction.

As a partial verification of the correctness of this formula, I remark that when $\phi = 0$ it becomes

$$3\mu\rho = 3\rho, \text{ or } \mu = v,$$

as it should.

Since (25) gives one and only one value of μ for each value of ϕ, no two being the same, as ϕ varies from 0 to 2π, $\mu\rho$ will perform a complete rotation about the axis v and return to its original position.

We may now write

$$cy^{\frac{3\phi}{2\pi}}\rho = \mu\rho, \tag{26}$$

where μ is a known vector.

In a similar manner we may compute the vector factor which will produce a given rotation about any given axis.

In fact, denoting any vector axis of length $\sqrt{3}$ by $\alpha + \beta + \gamma$, where α, β and γ are three rectangular unit vectors so situated that the given axis is the diagonal of a cube of which α, β, γ are the edges, any vector whatever may be denoted by

$$\rho = x\alpha + y\beta + z\gamma, \tag{27}$$

and the problem is to find a substitution μ which rotates ρ through an angle ϑ about v'.

The equations of condition are

$$S.\mu\rho v' = S.\rho v', \tag{28}$$

$$S.\mu^2\rho^2 = S.\rho^2, \tag{29}$$

$$S.\mu\rho^2 - \tfrac{1}{3}S^2.v\mu\rho = \cos\vartheta\,(S.\rho^2 - \tfrac{1}{3}S^2.v'\rho),$$

or

$$S.\mu\rho^2 = \tfrac{1}{3}\left[\cos\vartheta\,(3S.\rho^2 - S^2.v'\rho) + S^2.v'\rho\right]. \tag{30}$$

Equation (30) is obtained as follows:

The component of ρ parallel to v' being tv', we shall have $S.v'(\rho - tv') = 0$, whence

$$tS.v'^2 = S.v'\rho,$$

or

$$t = \tfrac{1}{3}S.v'\rho.$$

Hence the component of ρ perpendicular to v' is

$$\rho - \tfrac{1}{3} v' S.v'\rho, \tag{31}$$

and that of $\mu\rho$ is

$$\mu\rho - \tfrac{1}{3} v' S.v'\mu\rho. \tag{32}$$

The vectors (31) and (32) have the same length. Observing that $S.v'^2 = 3$, and that the scalar of their product divided by the square of the length of either gives the cosine of the angle between them, we have

$$\cos\vartheta = \frac{S.\mu\rho^2 - \frac{1}{3} S^2.v'\mu\rho}{S.\rho^2 - \frac{1}{3} S^2.v'\rho},$$

from which equation (30) follows at once.

Proceeding as before we shall obtain three equations corresponding to (9), (10) and (11) respectively,

$$S.\nu\mu\rho^2 = \tfrac{1}{3}\left[\cos(\vartheta + \phi)(3S.\rho^2 - S^2.v'\rho) + S^2.v'\rho\right], \tag{33}$$

$$S.\nu\mu^2\rho^2 = \tfrac{1}{3}\left[\cos\phi\,(3S.\rho^2 - S^2.v'\rho) + S^2.v'\rho\right], \tag{34}$$

$$S.v'\mu\nu\rho = S.v'\rho. \tag{35}$$

Using the formula (2) to expand ν in terms of $\mu\rho^2$, $\mu^2\rho^2$, and $v'\mu\rho$, we obtain the equation

$$\nu S.v'\mu\rho\, V(\mu\rho^2, \mu^2\rho^2) = V(\mu^2\rho^2, \mu\rho v')\, S.\nu\mu\rho^2 + V(\mu\rho v', \mu\rho^2)\, S.\nu\mu^2\rho^2 + V(\mu\rho^2, \mu^2\rho^2)\, S.v'\nu\mu\rho, \tag{36}$$

which, by aid of (13), becomes

$$\nu S.v'\mu\rho cy\,(\mu\rho^2)\, cy^2\,(\mu\rho^2)\, V(v, \mu) = cy\,(\mu\rho)\, cy^2\,(\mu\rho)\, V(\mu\rho, v')\, S.\nu\mu\rho^2 + cy\,(\mu\rho)\, cy^2\,(\mu\rho)\, V(v', \rho)\, S.\nu\mu^2\rho^2 + cy\,(\mu\rho^2)\, cy^2\,(\mu\rho^2)\, V(v, \mu)\, S.v'\nu\mu\rho. \tag{37}$$

From this we may divide out the factor $cy\,(\mu\rho)\, cy^2\,(\mu\rho)$, leaving

$$\left.\begin{aligned} &\nu\mu\rho S.v' cy\rho cy^2\rho\, V(v, \mu) \\ &\quad = V(\mu\rho, v')\, S.\nu\mu\rho^2 + V(v', \rho)\, S.\nu\mu^2\rho^2 + cy\rho cy^2\rho\, V(v, \mu)\, S.v'\nu\mu\rho \\ &\quad = \tfrac{1}{3} S^2.v'\rho.\left[V(\mu\rho, v') + V(v', \rho)\right] \\ &\quad + \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho).\left[V(\mu\rho, v).\cos(\vartheta + \phi) + V(v', \rho)\cos\phi\right] \\ &\quad + S.v'\rho.cy\rho cy^2\rho\, V(v, \mu) \end{aligned}\right\} \tag{38}$$

by aid of equations (33), (34) and (35).

Equation (38) may be written

$$\nu\mu\rho S.v' V(\rho, \mu\rho) = \tfrac{1}{3} S^2.v'\rho.V(v', \rho - \mu\rho) + \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho)\left[V(\mu\rho, v')\cos(\vartheta + \phi) + V(v', \rho)\cos\phi\right] + V(\rho, \mu\rho)\, S.v'\rho.$$

If in this we assign to μ the value $\frac{cy'\rho}{\rho}$ and to ϑ the corresponding value of $\frac{2\pi}{3}$, it becomes

$$\left.\begin{aligned}\nu cy'\rho S.v'V(\rho,\ cy'\rho) &= \tfrac{1}{3}S^2.v'\rho.V(v',\ \rho - cy'\rho)\\ &+ \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho)\Big[V(cy'\rho,\ v')\cos\Big(\frac{2\pi}{3}+\phi\Big) + V(v',\ \rho)\cos\phi\Big]\\ &+ V(\rho,\ cy'\rho)\,S.v'\rho.\end{aligned}\right\}(39)$$

The vector ν turns $cy'\rho$ through the angle ϕ around the axis v'; hence changing $cy'\rho$ to ρ and therefore $cy'^2\rho$ to $cy'\rho$ and ρ to $cy'^2\rho$, we shall have the vector μ sought for:

$$\left.\begin{aligned}\mu\rho S.v'V(cy'^2\rho,\ \rho) &= \tfrac{1}{3}S^2.v'\rho.V(v',\ cy'^2\rho - \rho)\\ &+ \tfrac{1}{3}(3S.\rho^2 - S^2.v'\rho)\Big[V(\rho,\ v')\cos\Big(\frac{2\pi}{3}+\phi\Big) + V(v',\ cy'^2\rho)\cos\phi\Big]\\ &+ V(cy'^2\rho,\ \rho)\,S.v'\rho.\end{aligned}\right\}(40)$$

Now $cy'^2\rho = x\beta + y\gamma + z\alpha$ and $V(cy'^2\rho,\ \rho)$

$$\begin{aligned}&= (xcy\beta + ycy\gamma + zcy\alpha)(xcy^2\alpha + ycy^2\beta + zcy^2\gamma)\\ &\qquad\qquad - (xcy^2\beta + ycy^2\gamma + zcy^2\alpha)(xcy\alpha + ycy\beta + zcy\alpha)\\ &= (yz - x^2)V(\alpha,\ \beta) + (zx - y^2)V(\beta,\ \gamma) + (xy - z^2)V(\gamma,\ \alpha)\\ &= (zx - y^2)\alpha + (xy - z^2)\beta + (yz - x^2)\gamma,\end{aligned}$$

owing to the property of three rectangular unit vectors that

$$V(\alpha,\ \beta) = \gamma;\quad V(\beta,\ \gamma) = \alpha;\quad V(\gamma,\ \alpha) = \beta.^*$$

We conclude that

$$S.v'V(cy'^2\rho,\ \rho) = (zx + xy + yz) - (x^2 + y^2 + z^2) = -\tfrac{1}{2}(3S.\rho^2 - S^2.v'\rho).$$

Also

$$\begin{aligned}V(v',\ cy'^2\rho - \rho) &= V(v',\ (z - x)\alpha + (x - y)\beta + (y - z)\gamma)\\ &= (cy\alpha + cy\beta + cy\gamma)((z - x)cy^2\alpha + (x - y)cy^2\beta + (y - z)cy^2\gamma)\\ &\quad - (cy^2\alpha + cy^2\beta + cy^2\gamma)((z - x)cy\alpha + (x - y)cy\beta + (y - z)cy\gamma),\end{aligned}$$

which reduces to $(2y - x - z)\alpha + (2z - x - y)\beta + (2x - y - z)\gamma$. Hence

$$\tfrac{1}{3}S^2.v'\rho.V(v',\ cy'^2\rho - \rho) + S.v'\rho.V(cy'^2\rho,\ \rho) = -\frac{1}{6}(3S.\rho^2 - S^2.v'\rho)\,S.v'\rho.$$

Dividing out the common factor v', equation (36) reduces to

$$\tfrac{1}{2}\mu\rho = -\tfrac{1}{3}\Big[V(\rho,\ v')\cos\Big(\frac{2\pi}{3}+\phi\Big) + V(v',\ cy'^2\rho)\cos\phi\Big] + \frac{1}{6}v'S.v'\rho,$$

* Cf. Tait's Quaternions, Art. 66.

or $$3\mu\rho = v'S.v'\rho - 2\left[V(\rho, v')\cos\left(\frac{2\pi}{3} + \phi\right) + V(v', cy'^2\rho)\cos\phi\right], \tag{41}$$

a formula which is very similar to (22), to which it reduces when $v' = v$.

When $\phi = 90°$ and ρ is perpendicular to v', equation (38) becomes

$$3\mu\rho = \sqrt{3}\,V(\rho, v'), \tag{42}$$

as it should.

Taking $S.v'\rho = 0$, we have now

$$3cy'^{\frac{3\vartheta}{2\pi}}\rho = V(v', \rho)\cos\left(\frac{2\pi}{3} + \vartheta\right) + V(cy'^2\rho, v')\cos\vartheta; \tag{43}$$

and differentiating this with respect to ϑ, we find that

$$\left.\begin{aligned} d.3cy'^{\frac{3\vartheta}{2\pi}}\rho &= -V(v', \rho)\sin\left(\frac{2\pi}{3} + \vartheta\right)d\vartheta - V(cy'^2\rho, v')\sin\vartheta d\vartheta \\ &= 3cy'^{\frac{3\left(\vartheta + \frac{\pi}{2}\right)}{2\pi}}\rho.d\vartheta. \end{aligned}\right\} \tag{44}$$

Writing for brevity $cy'^{\frac{3\vartheta}{2\pi}} = q^\vartheta$, so that $q\rho^{2\pi} = cy'^3\rho = \rho$, and observing that $V(v', \rho) = xq^{-\frac{\pi}{2}}\rho$, and $V(cy'^2\rho, v') = V(q^{\frac{4\pi}{3}}\rho, v') = -q^{-\frac{\pi}{2}}q^{\frac{4\pi}{3}}\rho = -q^{\frac{5\pi}{6}}\rho$ we can write equations (39) and (40) in the form

$$3q^\vartheta\rho = q^{-\frac{\pi}{2}}\rho\cos\left(\frac{2\pi}{3} + \vartheta\right) - q^{\frac{5\pi}{6}}\rho\cos\vartheta \tag{45}$$

and $$d.q^\vartheta\rho = q^{\vartheta + \frac{\pi}{2}}.\rho d\vartheta. \tag{46}$$

We now perceive that taking α a unit vector perpendicular to v', the equation of a plane curve may be written

$$\rho = rq^\vartheta\alpha \tag{47}$$

where r is a scalar. Differentiating twice in succession with respect to the time, we find that

$$\rho' = r'q^\vartheta\alpha + rq^{\vartheta + \frac{\pi}{2}}\alpha.\frac{d\vartheta}{dt}, \tag{48}$$

$$\rho'' = \left(r'' - r\left(\frac{d\vartheta}{dt}\right)^2\right)q^\vartheta\alpha + \left(2r'\frac{d\vartheta}{dt} + r\frac{d^2\vartheta}{dt^2}\right)q^{\vartheta + \frac{\pi}{2}}\alpha. \tag{49}$$

These equations give the velocity and acceleration of a body moving in a plane resolved along and perpendicular to the radius vector.*

* Cf. Tait's Quat., Art. 337.

Having thus, by a succession of operations comprised well within the domain of algebra, and with the aid of notations borrowed from Quaternions, obtained an operator which rotates a vector through a definite angle about a definite axis, I will now pause to compare q^{ϑ} with the unit versor $\varepsilon^{\frac{2\vartheta}{\pi}}$ of quaternions.*

$\varepsilon^{\frac{2\vartheta}{\pi}}$ and q^{ϑ} have precisely the same effect on a vector in the plane perpendicular to their axes; but $\varepsilon^{\frac{2\vartheta}{\pi}}$ operating on ε gives $\varepsilon^{\frac{2\vartheta+\pi}{\pi}}$, a versor whose angle exceeds that of $e^{\frac{2\vartheta}{\pi}}$ by $\frac{\pi}{2}$, its plane being the same; while q^{ϑ} operating on its axis v' simply leaves it unchanged. For this reason the effects of these operators on vectors not perpendicular to their axes are different, that of q^{ϑ} being the simpler. We may also compare q^{ϑ} with the quaternion operator $q.\alpha q^{-1}$ which rotates α about the axis and through double the angle of q.†

Returning to equation (41) and observing that if

$$\delta = a\alpha + b\beta + c\gamma, \quad \delta' = a'\alpha + b'\beta + c'\gamma, \tag{50}$$

then $$V(\delta, \delta') = (bc' - cb')\,\alpha + (ca' - ac')\,\beta + (ab' - ba')\,\gamma,$$

so that

$$V(\rho + 2cy'^2\rho, v') = (2x - y - z)\,\alpha + (2y - z - x)\,\beta + (2z - y - x)\,\alpha;$$

and writing the equation

$$3\mu\rho = v'S.v'\rho + \cos\phi\, V(\rho + 2cy'^2\rho, v') + \sqrt{3}\sin\phi\, V(\rho, v'), \tag{51}$$

we have, if ϕ is a very small angle, and if we note that $v'.Sv'\rho = (x + y + z)$ $(\alpha + \beta + \gamma)$,

$$3\mu\rho = 3\rho + \sqrt{3}\,d\phi\, V(\rho, v'), \tag{52}$$

to quantities of the second order.

For a rotation of σ through an angle $d\vartheta$ about an axis v'', which meets v', we have $$3\mu'\sigma = 3\sigma + \sqrt{3}\,d\vartheta\, V(\sigma, v'').$$

In this writing $\mu\rho$ for σ, we shall have the effect of the two rotations in succession upon ρ. This gives

$$\left.\begin{aligned} 9\mu'\mu\rho &= 3\,(3\rho + \sqrt{3}\,d\phi\, V(\rho, v') + \sqrt{3}\,d\vartheta\, V(3\rho + \sqrt{3}\,d\phi\, V(\rho, v'), v'') \\ &= 9\rho + 3\sqrt{3}\,d\phi\, V(\rho, v') + 3\sqrt{3}\,d\vartheta\, V(\rho, v'')\ddagger \\ &= 9q^{d\vartheta} q^{d\phi}\rho. \end{aligned}\right\} \tag{53}$$

Here the order of the rotations is indifferent, but that is of course not true for finite rotations, μ' being in general a function of $\mu\rho$.

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*Tait, p. 88; Ex. 13. † Tait, p. 261. ‡ Compare Tait, p. 260.

4.—*Applications to Trigonometry.*

The determinant

$$\begin{vmatrix} cy\alpha & cy\alpha & cy^2\alpha \\ cy\beta & cy\beta & cy^2\beta \\ cy\gamma & cy\gamma & cy^2\gamma \end{vmatrix}$$

is identically zero. Hence we conclude that for any three vectors whatever

$$cy\alpha\, V(\beta,\gamma) + cy\beta\, V(\gamma,\alpha) + cy\gamma\, V(\alpha,\beta) = 0.$$

If α, β, γ lie in a plane and their lengths are a, b, c and the angles between them A, B and C, then $V(\beta,\gamma) = bcv' \sin A$, $V(\gamma,\alpha) = cav' \sin \beta$, $V(\alpha,\beta) = abv' \sin C$, v' being the direction of the perpendicular on the plane, and equation (50) becomes, after dividing by v',

$$bc \sin A\, cy\alpha + ca \sin B\, cy\beta + ab \sin C\, cy\gamma = 0;$$

operating on this with cy^2, we have, after dividing by abc, and denoting the versors* of α, β, γ by α', β', γ',

$$\alpha' \sin A + \beta' \sin B + \gamma' \sin C = 0, \tag{55}$$

which shows that if the sides are proportional to the sines of the opposite angles, the plane figure will be closed and a triangle. If $\alpha + \beta + \gamma = 0$ correspond to any other closed figure formed by multiples of α', β', γ', it must be a consequence of (51); otherwise we could eliminate either vector and must conclude that the remaining two are parallel.†

Again, from the equation

$$-\alpha = \beta + \gamma, \tag{56}$$

we have, by squaring and taking scalars,

$$S.\alpha^2 = S.\beta^2 + S.\gamma^2 + 2S.\beta\gamma, \tag{57}$$

or

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{58}$$

The use of the negative sign in the last term is necessary from the manner of measuring the angle A.

With a spherical triangle we may proceed as follows: We have, if α, β, γ are unit vectors to the vertices,

$$S.V(\alpha,\beta)\, V(\beta,\gamma) = -\cos B \sin a \sin c.$$

*Tait, p. 33.

† Cf. Gibbs, Vector Analysis, p. 49.

Now

$$V(\alpha,\ \beta)\,V(\beta,\ \gamma) = (cy\alpha cy^2\beta - cy\beta cy^2\alpha)(cy\beta cy^2\gamma - cy\gamma cy^2\beta)$$
$$= -\,cy\beta^2 cy^2(\alpha\gamma) + cy(\beta\gamma)\,cy^2(\alpha\beta) + cy(\alpha\beta)\,cy^2(\beta\gamma) - cy(\alpha\gamma)\,cy^2\beta^2;$$

and recalling that

$$S.cy(\alpha\beta)\,cy^2(\beta\gamma) = S.\beta\gamma cy^2(\alpha\beta) \text{ and } S.cy(\beta\gamma)\,cy^2(\alpha\beta) = S.\beta\gamma cy(\alpha\beta),$$

by a formula of Art. 2, we may write this

$$S.V(\alpha,\ \beta)\,V(\beta,\ \gamma) = S.\beta\gamma\,(cy(\alpha\beta) + cy^2(\alpha\beta)) - S.\beta^2.(cy(\alpha\gamma) + cy^2(\alpha\gamma))$$
$$= S.\beta\gamma\,(vS.\alpha\beta - \alpha\beta) - S.\beta^2\,(vS.\alpha\gamma - \alpha\gamma)$$
$$= S.\beta\gamma S.\alpha\beta - S.\beta^2 S.\alpha\gamma$$
$$= \cos a \cos c - \cos b,$$

by previous formulae. We conclude that

$$\cos b = \cos a \cos c + \sin a \sin c \cos B.^*$$

This extremely simple process may be compared with the analysis given in Tait, p. 56, Art. 90, which must be read to get the result on p. 71.

Without following so closely the familiar processes of quaternions, we may proceed as follows:

$$V(\alpha,\ \beta)\,V(\beta,\ \gamma) = \begin{vmatrix} cy\alpha & cy\beta \\ cy^2\alpha & cy^2\beta \end{vmatrix} . \begin{vmatrix} cy\beta & cy\gamma \\ cy^2\beta & cy^2\gamma \end{vmatrix}$$
$$= \begin{vmatrix} cy(\alpha\beta) + cy^2(\alpha\beta), & cy(\alpha\gamma) + cy^2(\alpha\gamma) \\ cy\beta^2 + cy^2\beta^2 \quad , & cy(\beta\gamma) + cy^2(\beta\gamma) \end{vmatrix}$$
$$= \begin{vmatrix} vS.\alpha\beta - \alpha\beta, & vS.\alpha\gamma - \alpha\gamma \\ vS.\beta^2 - \beta^2 \quad , & vS.\beta\gamma - \beta\gamma \end{vmatrix}$$
$$= vS.\alpha\beta S.\beta\gamma - \beta\gamma S.\alpha\beta - \alpha\beta S.\beta\gamma - vS.\beta^2 S.\alpha\gamma$$
$$+ \alpha\gamma S.\beta^2 + \beta^2 S.\alpha\gamma.$$

Hence, taking scalars, we have

$$S.\alpha\beta S.\beta\gamma - S.\beta^2 S.\alpha\gamma = S.V(\alpha,\ \beta)\,V(\beta,\ \gamma)$$

as before, by remembering that $S.v = 3$.

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* Tait, p. 71.